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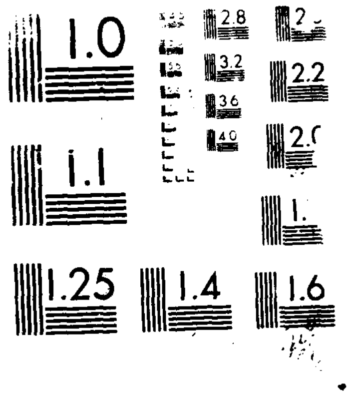
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LOGNORMAL DISTRIBUTION MAXIMUM-LIKELIHOOD PARAMETER ESTIMATION ALGORITHMS

H. P. Dudel
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MARCH 1989

U.S. ARMY MISSILE COMMAND
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Microcomputer-based algorithms for lognormal distribution parameter estimation are presented. A program diskette will be made available upon request. For given data, the parameters (three or two) are estimated (1) by means of the moments, and (2) by means of the maximum-likelihood principle. Maximum-likelihood estimation is done in two different ways: (1) by means of the derivative equations which result from the logarithmic likelihood function, and (2) by means of direct optimization of the logarithmic likelihood function itself. Moment estimates are used as starting values for the optimization process.

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SUMMARY

Microcomputer-based algorithms for lognormal distribution parameter estimation are presented. A program diskette will be made available upon request. For given data the parameters (three or two) are estimated (i) by means of the moments, and (ii) by means of the maximum-likelihood principle. Maximum-likelihood estimation is done in two different ways: (i) by means of the derivative equations which result from the logarithmic likelihood function, and (ii) by means of direct optimization of the logarithmic likelihood function itself. Moment estimates are used as starting values for the optimization process.

I. INTRODUCTION

The lognormal probability distribution class is defined by its probability density function (PDF)

$$f(x) = \begin{cases} \left[\sigma(x-\lambda)\sqrt{2\pi}\right]^{-1} \exp - \left(2\sigma^2\right)^{-1} \left[\log(x-\lambda) - \mu\right]^2, x > \lambda, \\ 0, x \leq \lambda. \end{cases} \quad (*)$$

The distribution parameters are: shift $\lambda \in R$, shape $\sigma > 0$, and scale $\mu \in R$. The scaling property of μ becomes evident under the transformation $\mu = \log \gamma$, $\gamma > 0$, so that $\log(x-\lambda) - \mu = \log[(x-\lambda)\gamma^{-1}]$.

The lognormal PDF (*) is unimodal. It takes exactly one maximum in the internal (λ, ∞) which is located at $x = \lambda + \exp(\mu-\sigma^2)$ (mode). It is of diffusion type, i.e., it is generated by a one-dimensional parabolic differential equation of Fokker-Planck type.

Two algorithms for estimation of the three parameters of (*) relative to given statistical data are presented which are based on the maximum-likelihood (ML) principle. Both can be used for sample (raw) data as well as histogram (grouped) data. One of them utilizes the three equations which are obtained by equating to zero the three first partial derivatives of the associated logarithmic likelihood function (LLF). Two of the parameters (shape and scale) can be eliminated so that only one equation, $h(\lambda) = 0$, actually remains to be solved. The analytical background of this algorithm is not new. The best account of it may be found in [1]. The numerical, microcomputer oriented, algorithmic approach presented in this report is new, however, and highly efficient.

The other algorithm is based on direct optimization of the LLF associated with the PDF (*). It may be somewhat slower to execute than the other one, but it has the advantage of avoiding the problems created by multiple roots of the equation $h(\lambda) = 0$. As a matter of fact, if the estimation problem has a solution at all, $h(\lambda)$ has at least two distinct zeros.

In many practical situations, the shift parameter is known to have a specific value λ^*. In this situation the general three-parameter problem reduces to a two-parameter one. It is easily solved by means of the ML derivative equations approach. The ML optimization algorithm in the two-parameter case uses a LLF which, for programming efficiency, is maintained as a function of three variables in an artificial way (see Sec. VIII.4).

Moment estimates of the parameters (two or three depending on whether the shift parameter is known or not) are easily obtained. They are used as starting values for the ML optimization algorithm.

Questions of hypothesis justification and goodness of fit are outside the framework of this report and, thus, are not addressed here.

A program diskette will be made available upon request.

II. THE DIFFUSION CHARACTER OF THE LOGNORMAL DISTRIBUTION

It is well known [2], [3] that the function

$$f(x,t) = \left(\sqrt{\pi}\, b\right)^{-1} \exp - \left(b^{-1} x\right)^2 , \quad x \in R , \tag{II.1}$$

with

$$b = b(t) = \begin{cases} \left[4\pi t\right]^{1/2} , & \tau = 0 , \\ \left[2\alpha\tau^{-1} (1 - \exp - 2\tau t)\right]^{1/2} , & \tau \neq 0 , \end{cases} \tag{II.2}$$

is the delta function initial condition (at $x = 0$, $t = 0$) solution of the autonomous Fokker-Planck equation

$$\left.\begin{aligned} &\left[A(x)\, z(x,t)\right]_{xx} \quad \left[D(x)\, z(x,t)\right]_x - z_t(x,t) = 0 , \quad x \in R , \; t > 0 , \\ &A(x) = \alpha > 0 \qquad \text{(diffusion coefficient)} , \\ &D(x) = -\tau x \qquad \text{(drift coefficient)} . \end{aligned}\right\} \tag{II.3}$$

The subscripts in (II.3) signify partial derivatives. For $\tau = 0$, (II.3) reduces to the standard heat equation.

With t in (II.2) considered as a parameter the function (II.1) becomes the normal (Gauss) PDF with scale parameter b .

The transformation $x = \log \gamma^{-1} y$ generates from (II.1), after y has been replaced by x, the function

$$g(x,t) = f\left(\log \gamma^{-1} x, t\right) \left|\frac{dx}{dy}\right| = \left(\sqrt{\pi}\, bx\right)^{-1} \exp - b^{-2} \log^2 \gamma^{-1} x \tag{II.4}$$

with $b = b(t)$ given by (II.2). (For details on this kind of PDF transformations the reader is referred to [4,(5.6)].) For any fixed $t > 0$ and with $b = \sqrt{2}\,\sigma$, $\log \gamma = \mu$, this function takes the form (*) with $\lambda = 0$. The shift parameter λ in (*) is immaterial in this context.

The same transformation applied to the differential equation (II.3) leads to the autonomous Fokker-Planck equation

$$\left.\begin{aligned} &\left[A^*(x)\, w(x,t)\right]_{xx} - \left[D^*(x)\, w(x,t)\right]_x - w_t(x,t) = 0 \,, \\ &A^*(x) = \alpha x^2 \,, \\ &D^*(x) = \left[\alpha - \tau \log \gamma^{-1} x\right] x \,, \end{aligned}\right\} \tag{II.5}$$

of which the function $g(x,t)$ defined in (II.4) is a solution. As a matter of fact, the function

$$v^*(x,t;y) = \left(\sqrt{\pi}\, bx\right)^{-1} \exp - b^{-2} \left[\log \gamma^{-1} x - e^{-\tau t} \log \gamma^{-1} y\right]^2$$

with $b = b(t)$ again defined by (II.2), is the delta function initia' condition (at $x = y$, $t = 0$) solution of (II.5). Relative to applications, for example in nucleation evolution processes [5, Secs. 5.3.1, 5.3.3], it is of interest to note that the scale parameter

$$\mu = \log \left[\gamma \left(\gamma^{-1} y\right) \exp - \tau t\right]$$

depends on time t unless the drift parameter τ is zero.

III. DATA SETS

In population estimation problems the parameter vector $P = (\lambda, \sigma, \mu)$ of the lognormal distribution class (*) is to be specified from a given set of N observations. (In general terms, the observations represent not necessarily distinct i.i.d. sample values of a random variable which is assumed to be distributed according to some PDF.)

It may be assumed that the given observations form a set of $M \leq N$ distinct elements $x_1 < x_2 < \ldots < x_M$, located on a coordinate axis of x, each of absolute frequency (integral multiplicity) $f_{a\nu} \geq 1$, $N = \sum_{\nu=1}^{M} f_{a\nu} > 1$.

In histogram estimation problems, the parameter vector $P = (\lambda, \sigma, \mu)$ of (*) is to be specified from a given set of absolute frequencies $f_{a\nu} \geq 0$ associated with adjacent class intervals of the form $[a+(\nu-1)\Delta a,\ a + \nu\Delta a)$ ($\nu=1, \ldots, M$), $a \in R$, $\Delta a > 0$, of equal length Δa, located on a coordinate axis of x. Actually, the vector P is to be specified form the set $\{(x_\nu, f_{a\nu})\}$ of the coordinate pairs $(x_\nu, f_{a\nu})$ with x_ν taken as the midpoint of the class interval with number ν, i.e., $x_\nu = a + (\nu-1/2)\Delta a$. In this sense the histogram estimation problem is a curve fitting problem.

There is a variant of the histogram estimation problem which is based on given data sets $\{(x_\nu, f_{a\nu})\}$ with arbitrarily spaced observations $x_1 < x_2 < \ldots < x_M$.

It is convenient to have a uniform notation to cover all of these estimation problems. Therefore, it will be assumed that the given data are of the general form of a set of ordered pairs $\{(x_\nu, f_{a\nu})\}$ ($\nu=1, \ldots M$) with $x_1 < x_2 < \ldots < x_M$, and $M \leq N = \sum_{\nu=1}^{M} f_{a\nu}$. In population estimation problems $M \leq N$, $f_{a\nu} \geq 1$, whereas in histogram estimation problems $M < N$, $f_{a\nu} \geq 0$ ($\nu=2, \ldots, M-1$), $f_{a1} \geq 1$, $f_{aM} \geq 1$, and $x_\nu = a + (\nu-1/2)\Delta a$.

The shift parameter λ, in either case, now is necessarily restricted to the half-interval $\lambda < x_1$.

For the analytical description of the estimation problem it is also convenient to introduce the relative frequencies $f_\nu = N^{-1} f_{a\nu}$, $\sum_{\nu=1}^{M} f_\nu = 1$.

It may happen that a value λ^* is known for the shift parameter, either from physical or other evidence. This case, which reduces the three-parameter problem to a two-parameter one with only σ and μ unknown, is incorporated in the general estimation algorithms.

IV. THE MOMENTS

The first three population moments for the PDF(*) are given by

$$M_1 = \int_\lambda^\infty x f(x)\,dx \qquad = \lambda + e^{\mu}\omega^{1/2} \ ,$$

$$M_2 = \int_\lambda^\infty \left(x - M_1\right)^2 f(x)\,dx = e^{2\mu}\omega(\omega-1) \ ,$$

$$M_3 = \int_\lambda^\infty \left(x - M_1\right)^3 f(x)\,dx = e^{3\mu}\omega^{3/2}(\omega+2)(\omega-1)^2 \ , \ \omega = \exp\sigma^2 \ .$$

$M_2 > 0$, $M_3 > 0$ since $\omega > 1$.

The corresponding sample moments are, with the notation conventions introduced in Section III,

$$m_1 = \bar{x} = \sum_{\nu=1}^{M} x_\nu f_\nu \ ,$$

$$m_2 = \sum_{\nu=1}^{M} \left(x_\nu - \bar{x}\right)^2 f_\nu \ ,$$

$$m_3 = \sum_{\nu=1}^{M} \left(x_\nu - \bar{x}\right)^3 f_\nu \ .$$

Also of interest are the moment coefficients of skewness,

$$\alpha_3(\omega) = M_2^{-3/2} M_3 = (\omega+2)(\omega-1)^{1/2}$$

and

$$a_3 = m_2^{-3/2} m_3 \ .$$

It should be observed that the number a_3 may turn out to be nonpositive for a given data set whereas α_3 , which has been derived from the theoretical moments, is

always positive. Therefore, if $a_3 \leq 0$, moment estimation of the lognormal parameters is impossible. The reader is referred to [1, Sec. 5] for additional remarks.

Estimation of the three parameters λ, σ, and μ for the PDF (*) can now be accomplished as follows. Since α_3 is independent of μ and λ, the equation $a_3 = \alpha_3$, in which a_3 is a known number, leads to the equation

$$p(\omega) = \omega^2(\omega+3) - \left(4 + a_3^2\right) = 0 .$$

Application of Descartes' Rule of Signs [6, Lemma 6.2] shows that $p(\omega)$ has exactly one simple positive zero ω_0 which, obviously, is greater than unity. It provides the moment estimate for the shape parameter σ,

$$\sigma_0 = \left(\log \omega_0\right)^{1/2} . \qquad \text{(IV.1)}$$

The equation $m_2 = M_2$, then leads to

$$\mu_0 = \frac{1}{2}\left\{\log m_2 - \log\left[\omega_0\left(\omega_0 - 1\right)\right]\right\} . \qquad \text{(IV.2)}$$

Finally, the equation $m_1 = M_1$ gives

$$\lambda_0 = m_1 - \omega_0^{1/2} \exp \mu_0 .$$

Thus, the vector of the parameters estimated by means of the moments is $P_0 = (\lambda_0, \sigma_0, \mu_0)$.

If the value λ^* of the shift parameter λ is already known, estimates σ_0 and μ_0 of σ and μ can be obtained from the first two moments. Elimination of μ immediately leads to

$$\omega_0 = 1 + m_2\left(m_1 - \lambda^*\right)^2 .$$

With this value of ω_0 the moment estimates for σ and μ follow from (IV.1) and (IV.2), respectively.

V. THE LOGARITHMIC LIKELIHOOD FUNCTION

In general terms, but on the basis of the notational conventions adopted in Section III, the likelihood function of a random variable X for which there exists a data set $\{(x_\nu, f_{a\nu})\}$ $(\nu=1, \ldots, M)$ and which is assumed to be distributed according to a PDF $f(x;P)$, which depends on a parameter vector $P = (\lambda, p_1, p_2, \ldots)$, λ = shift, is defined by

$$L(P) = \prod_{\nu=1}^{M} \left[f\left(x_\nu - \lambda ; p_1, p_2, \ldots\right)\right]^{f_{a\nu}} = \prod_{\nu=1}^{M} \left[f\left(x_\nu - \lambda ; p_1, p_2, \ldots\right)\right]^{Nf_\nu}$$

(see, e.g., [7, Sec. 12.5], [8, Sec. 5.4].) For the lognormal PDF class (*) with $P = (\lambda, \sigma, \mu)$ and with the abbreviation

$$\log(x_\nu - \lambda) = \rho_\nu(\lambda) \quad , \quad \lambda < x_1 \quad ,$$

the function $L(P)$ takes the particular form

$$L(P) = (2\pi)^{-N/2} \exp - N \left[\log \sigma + \sum_{\nu=1}^{M} f_\nu \rho_\nu(\lambda) + \left(2\sigma^2\right)^{-1} \sum_{\nu=1}^{M} f_\nu \left(\rho_\nu(\lambda) - \mu\right)^2 \right] .$$

The function

$$\phi(P) = N^{-1} \log \left[(2\pi)^{N/2} L(P) \right]$$

$$= - \log \sigma - \sum_{\nu=1}^{M} f_\nu \rho_\nu(\lambda) - \left(2\sigma^2\right)^{-1} \sum_{\nu=1}^{M} f_\nu \left(\rho_\nu(\lambda) - \mu\right)^2 \qquad \text{(V.1)}$$

is called the logarithmic likelihood function (LLF) of the lognormal distribution class (*).

The problem now is to specify numerical values for the parameters λ, σ, and μ for which (*) approximates the given data optimally. The maximum-likelihood principle asserts that the optimal parameter values, if they exist, are the coordinates of the point $\hat{P} = (\hat{\lambda}, \hat{\sigma}, \hat{\mu})$ in the open parameter space $\mathcal{P} : \{\lambda < x_1 , \sigma > 0, \mu \in R\}$ at which the LLF $\phi(P)$ takes its global maximum. The openness of the space $\mathcal{P}$ is essential in this context as will be seen in Section VI.

VI. MAXIMUM-LIKELIHOOD PARAMETER ESTIMATION

The first approach is based on the derivative equations which result from (V.1). A necessary condition on the parameters is that they satisfy the three partial derivative equations $\partial\phi/\partial\lambda = 0$, $\partial\phi/\partial\sigma = 0$, and $\partial\phi/\partial\mu = 0$. They are of the form

$$\frac{\partial\phi}{\partial\lambda} = \sum_{\nu=1}^{M} f_\nu \exp - \rho_\nu + \sigma^{-2} \sum_{\nu=1}^{M} f_\nu \left(\rho_\nu - \mu\right) \exp - \rho_\nu = 0 \ , \tag{VI.1}$$

$$\frac{\partial\phi}{\partial\sigma} = - \sigma^{-1} + \sigma^{-3} \sum_{\nu=1}^{M} f_\nu \left(\rho_\nu - \mu\right)^2 = 0 \ , \tag{VI.2}$$

$$\frac{\partial\phi}{\partial\mu} = \sigma^{-2} \sum_{\nu=1}^{M} f_\nu \left(\rho_\nu - \mu\right) = 0 \ . \tag{VI.3}$$

Equations (VI.2) and (VI.3) can be written as

$$\sigma^2 = \sum_{\nu=1}^{M} f_\nu \left(\rho_\nu(\lambda) - \mu\right)^2 \ , \tag{VI.4}$$

$$\mu = \sum_{\nu=1}^{M} f_\nu \, \rho_\nu(\lambda) \ . \tag{VI.5}$$

If the shift parameter λ is known these two equations immediately represent the solution of the ML parameter estimation problem on the basis of the derivative equations. In this situation, equation (VI.1) becomes vacuous, and, for a given data set $\{(x_\nu, f_{a\nu})\}$ and known $\lambda = \lambda^*$, equation (IV.5) determines $\hat{\mu}$, and (VI.4), with $\mu = \hat{\mu}$, determines $\hat{\sigma}$ [8, Sec. 5.2.2, Table 5.1].

If the shift parameter λ is considered as unknown (and thus, together with σ and μ , to be estimated from the given data), equations (VI.4) and (VI.5) must be supplemented by equation (VI.1). However, by means of (VI.5) and (VI.4), μ and σ can be eliminated from (VI.1) so that only one equation in λ remains. It is convenient to perform this elimination in the LLF $\phi(P)$ given in (V.1) directly. To simplify the notation the following functions of λ are introduced:

$$A(\lambda) = \sum_{\nu=1}^{M} f_\nu \rho_\nu^2(\lambda) > 0 \quad ,$$

$$C(\lambda) = \sum_{\nu=1}^{M} f_\nu \rho_\nu(\lambda) = \mu \quad ,$$

$$E(\lambda) = -C'(\lambda) = \sum_{\nu=1}^{M} f_\nu \exp - \rho_\nu(\lambda) > 0 \ ,$$

$$F(\lambda) = -\frac{1}{2} A'(\lambda) = \sum_{\nu=1}^{M} f_\nu \rho_\nu(\lambda) \ \exp - \rho_\nu(\lambda) \ .$$

Then (VI.4) takes the form

$$\sigma^2 = A(\lambda) - C^2(\lambda) > 0 \ .$$

Positivity of this expression follows from Tchebychef's inequality [9, Thm. 43]. Elimination of μ and σ (by means of the expressions given above) from $\phi(\lambda,\sigma,\mu)$ results in the function

$$\Phi(\lambda) = -\frac{1}{2}\left[\log\left(A - C^2\right) + 2C + 1\right] .$$

For $\Phi(\lambda)$ to take a maximum, the derivative equation

$$\Phi'(\lambda) = \left(A - C^2\right)^{-1} h(\lambda) = 0$$

with

$$h(\lambda) = (A - C^2 - C)E + F$$

must be satisfied. Therefore, since $A - C^2 > 0$, the maximum-likelihood estimate λ_1 for the shift parameter λ must be a root of the equation

$$h(\lambda) = (A - C^2 - C)E + F = 0. \qquad \text{(VI.6)}$$

The following should now be observed. Since $C(\lambda) \downarrow -\infty$ as $\lambda \uparrow x_1 = \min\{x_\nu\}$, since $A(\lambda)C^{-2}(\lambda) \rightarrow f_1^{-1} > 1$ as $\lambda \uparrow x_1$, and since $\log|C(\lambda)|$ increases more slowly than $|C(\lambda)|$, the rewritten version

$$- \frac{1}{2}\left[2 \log |C(\lambda)| + \log\left(A(\lambda)C^{-2}(\lambda) - 1\right) + 2C + 1\right]$$

for $\Phi(\lambda)$ shows that $\Phi(\lambda) \uparrow \infty$ as $\lambda \uparrow x_1$. As a consequence of this result it is seen that the openness of the parameter space $\mathcal{P} : \{\lambda < x_1 , \sigma > 0, \mu \in R\}$ is essential for the identification of the location of the global maximum of $\phi(P)$ in the interior of $\mathcal{P}$ (see the last paragraph of Section V). Another consequence is that, if $\Phi(\lambda)$ takes a maximum at all, it must, in the direction of decreasing λ, take a minimum first. Consequently, if the equation (VI.6) has a root at all, it has at least two. The correct root relative to the estimation problem is the second one in the direction of decreasing λ.

If λ_1 is the correct root of (VI.6) the estimates μ_1 and σ_1 for μ and σ are obtained as

$$\mu_1 = C(\lambda_1) \; , \; \sigma_1 = \left[A(\lambda_1) - C^2(\lambda_1)\right]^{1/2} ,$$

respectively.

The second ML parameter estimation procedure for the lognormal distribution class is based on direct optimization in the parameter space $\mathcal{P}$ of the LLF $\phi(P)$ given in (V.1). In optimization procedures it is customary to minimize the objective function. Therefore, instead of maximizing $\phi(P)$, the function $-\phi(P)$ will be minimized. The resulting coordinates of the minimum of $-\phi(P)$ will be denoted by λ_2, σ_2, μ_2.

Since both ML parameter estimation algorithms are based on the LLF $\phi(P)$ it is obvious that the resulting parameter estimate vectors P_1 and P_2 should agree within the tolerances associated with the different numerical procedures. Agreement of the vectors P_1 and P_2 indicates successful execution of the ML estimation process.

In the numerical implementation of the optimization ML estimation approach the moment estimation vector $P_0 = (\lambda_0, \sigma_0, \mu_0)$ (see Section IV) is used as the starting vector for the iteration processes. If the value λ^* of λ is known, the starting vector is $P_0 = (\lambda^*, \sigma_0, \mu_0)$.

VII. USER'S GUIDE

This section provides information on some of the numerical aspects of the estimation algorithms. Information about their use is given on the diskette.

1. Preparation of Data

According to Section III, the data input for the algorithms is a set $\{(x_{\nu 1}\ f_{a\nu})\}$ ($\nu=1, \ldots, M$) of coordinate pairs, ordered with respect to the abscissa values such that $x_1 < x_2 < \ldots < x_M$. Unless the data are supplied in this particular form, a subroutine does the ordering and frequency determination for given population data. For histogram data, a subroutine calculates the x_ν's for given $a \in R$ and $\Delta a > 0$.

For computational reasons, it is convenient to translate the original x_ν values along the coordinate axis of x .

(a) Shift Parameter Unknown

The desired translation is accomplished by addition of $-x_1$ to each x_ν so that the new $x_1 = 0$. The advantage of this operation is to have a universal upper bound, namely 0, for the unknown (translated) shift parameter λ . The shape and scale parameters σ and μ are not affected. The algorithms calculate the parameter values λ, σ, and μ and return the true parameter values $\lambda_t = \lambda + x_1$, σ_1 and μ_1 relative to the original coordinate axis.

(b) Shift Parameter Known

Let λ^* be the known value of the shift parameter λ . The desired translation is accomplished now by addition of $-\lambda^*$ to each x_ν. As before, σ and μ are not affected. The algorithms calculate the values σ and μ with shift equal to zero, and return the true values
$\lambda_t = \lambda^*$, σ_1 and μ .

2. Moment Estimation

Moment estimation is straightforward according to the formulas given in Section IV, provided $a_3 > 0$. In the three-parameter case, the equation $p(\omega) = 0$ is solved to find its root $\omega_0 > 1$. This is done by means of root bracketing and a modified version of Brent's method [10, Chap. 7.3].

3. Derivative Equations ML Estimation

In the three-parameter case the equation $h(\lambda) = 0$ (VI.6) is solved. Bracketing of the desired root λ_1 is achieved by calculation of $h(\lambda_\nu)$ along the search sequence $\lambda_{\nu+1} = (1.618)\lambda_\nu$ ($\nu = 0, 1, \ldots$), with $\lambda_0 = -0.05$, until h changes from negative to positive values at two successive points. Once λ_1 has been bracketed the modified Brent's method is used to locate it.

With λ_1 calculated (or with λ^* known) the estimates μ_1 and σ_1 are calculated from (VI.5) and (VI.4), respectively.

4. Optimization ML Estimation

Optimization of the LLF (V.1) is accomplished by means of Powell's method [10, Chap. 10.5] in both the three- and two-parameter cases. The vector $P_0 = (\lambda_0, \sigma_0, \mu_0)$ (or $P_0 = (\lambda^*, \sigma_0, \mu_0)$) of the moment estimates is used to start the minimization routine on the function $-\phi(P)$. Once Powell's tolerance has been achieved the algorithm continues by minimizing the function

$$\Psi(P) = (\exp 10)\left[-1 + \exp -10\left(\phi(P) - \phi(\bar{P})\right)\right]$$

in which $\bar{P}$ is the final vector from Powell's routine on $-\phi(P)$. Use of the "amplifier" function $\psi(P)$ eliminates the effects of shallowness of $-\phi(P)$ near its minimum and increases the accuracy of the numerical values of the coordinates of the minimum of $-\phi(P)$.

If λ is known to have the value λ^*, the LLF ϕ reduces to a function of only two variables. To avoid additional subroutines for this situation, $\phi(\lambda^*,\sigma,\mu)$ is maintained as a function of three variables, σ, μ, and λ, by addition of the term $(\lambda-\lambda^*)^2$.

5. Frequency Calculation

For histogram parameter estimation an auxiliary program is provided which calculates two sets of expected absolute frequencies. One set corresponds to the moment parameter estimates, the other to the ML estimates. (It should be remembered that, under successful execution of the two ML estimation algorithms, the derivative equations and the optimization estimates agree.) In both calculations the logarithm of the PDF(*) is used to obtain the relative frequencies at the points x_ν . They are subsequently converted into absolute frequencies.

To provide a basis for goodness-of-fit statements, chi-square values for the individual class intervals are also calculated.

VIII. EXAMPLES

To demonstrate the versatility of the algorithms presented in this report five examples are offered. They contain population and histogram estimates with shift parameter unknown or known. Accompanying tables are given in Section IX.

Example 1

N = 87 observations of annual 24-hour maximum rainfalls (in points) at Sidney, Australia, over the period 1859-1945 are given in Table 1.1 [11]. Moment and ML parameter estimates for the three unknown parameters are given in Table 1.2. The subscripts 0, 1, and 2 refer to moments, derivative equations, and optimization, respectively.

Grouping into histogram absolute frequency data has been performed on the data of Table 1.1 in five different ways, G_ν , as displayed in the second columns of Tables 1.4 G_ν (ν = 1, ... , 5). The first column shows the class interval numbers ν .

Table 1.3 contains the class interval data for the groupings G_ν (ν = 1, ... , 5).

The resulting parameter estimates for the grouped histogram data are shown in Tables 1.5 G_ν (ν = 1, ... , 5).

The estimated parameter values are used to calculate the expected absolute frequencies from the PDF(*). For the moment estimates they are given in the third columns of Tables 1.4 G_ν, and for the ML estimates in the fifth columns. Next to the calculated frequencies in Tables 1.4 G_ν are shown the chi-square values. As mentioned in the Introduction, no significance will be attached to them within the framework of this report.

Example 2

N = 20 random observations generated from a lognormal population with λ = 100, σ = 0.4, and μ = log 50 [1, Sec. 10] are displayed in Table 2.1. The three corresponding sets of estimated parameters are given in Table 2.2.

Since it is known how the data of Table 2.1 have been generated it is possible to process them under the assumption that the value λ^* = 100 for the shift parameter is known. The resulting two-parameter estimates are shown in Table 2.3.

This example is on the diskette as RANDSAM.DAT.

Example 3

This example deals with a total of N = 37 observations of frost-days in Munich in the months of April over the period 1930-1966 [12, Ex. 42, Table 31] which contain M = 18 distinct ones. The observations together with their frequencies are given in Table 3.1.

The resulting parameter estimates are shown in Table 3.2.

Example 4

A total of N = 885 observations of rainfall totals (in inch) for sets of four consecutive months at Camden Square, London, over the period 1870-1943 [13, Ex. 7.511] have been grouped into M = 17 class intervals $[a + (\nu-1)\Delta a, a + \nu\Delta a)$ with a = 2, $\Delta a = 1$. The data are given in Table 4.1.

The calculated parameter estimates are contained in Table 4.2.

The expected frequencies calculated from the parameters of Table 4.2 together with the corresponding chi-square values are shown in Table 4.1

It is reasonable to assume in the present example that the probability of no rainfall at the Camden Square location over the given period is zero. In other words, it is reasonable to assume that the shift parameter λ is known and that $\lambda = \lambda^* = 0$. The corresponding two-parameter estimates are given in Table 4.3.

Table 4.4 shows the expected frequencies for the two-parameter case.

This example is on the diskette as BROOKS.DAT.

Example 5

This example concerns weekly precipitation sums observations (in 10^{-2} inch) at Kwajalein, IS, for the summers 1949-1958 [14, p. 47, Ex. 10]. The total number of observations is N = 130. They have been grouped into M = 17 class intervals $[a + (\nu-1)\Delta a, a + \nu\Delta a)$ with a = 0, $\Delta a = 50$. Table 5.1 shows the given data. The estimated parameter values are displayed in Table 5.2. The corresponding frequencies are given in Table 5.1.

The estimates for σ and μ under the assumption that the shift parameter is zero are shown in Table 5.3. The expected frequencies in this case are displayed in Table 5.4.

IX. TABLES

TABLE 1.1. Annual Maximum Twenty-Four Hour Rainfalls (in Points) at Signey, Australia, 1859-1945

370	752	662	190	375	393	395	301
565	618	445	403	324	280	890	392
389	281	489	753	569	204	425	648
433	645	485	468	283	275	836	566
295	434	330	310	236	487	273	605
301	423	441	637	157	477	364	363
362	571	177	475	708	441	652	337
325	342	316	653	488	414	418	374
258	188	484	459	489	239	391	302
780	322	380	335	263	216	339	579
230	333	154	1105	330	192	420	

TABLE 1.2. Three-Parameter Estimates

```
LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is   RAINAUS.DAT
Number of parameters is   3

****************************************************************
THE MOMENT ESTIMATES FOR THE LOG-NORMAL DISTRIBUTION
Lambda(0)  =     -.722546E+002
Sigma(0)   =      .344057E+000
Mu(0)      =      .615229E+001

****************************************************************
THE DERIVATIVE ESTIMATES FOR THE LOG-NORMAL DIST
Lambda( 1)  =    -.394497E+001
Sigma( 1)   =     .400045E+000
Mu( 1)      =     .598452E+001

****************************************************************
THE POWELL ESTIMATES FOR THE LOG-NORMAL DIST
Lambda( 2)  =    -.389397E+001
Sigma( 2)   =     .400101E+000
Mu( 2)      =     .598437E+001

****************************************************************
```

TABLE 1.3. Class Intervals and Numbers of Classes for Groupings G_v ($v=1, \ldots ,5$)

	G_1	G_2	G_3	G_4	G_5
a	150	125	150	100	125
Δa	100	100	50	75	75
M	10	10	20	14	14

TABLE 1.4G_1. Grouped Frequency Data for Grouping G_1

LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is lngdset1.dat

		Moment Estimate		Max Likelihood Est	
v	f(abs)	f(cal)	X**2	f(cal)	X**2
1	11	10.64	.01	10.83	.00
2	23	21.05	.18	24.62	.11
3	23	20.97	.20	20.95	.20
4	10	14.92	1.63	13.38	.85
5	10	8.88	.14	7.72	.68
6	4	4.77	.12	4.30	.02
7	4	2.42	1.02	2.38	1.11
8	1	1.19	.03	1.32	.08
9	0	.58	.58	.75	.75
10	1	.28	1.87	.43	.77
Totals	87	85.71		86.68	

TABLE 1.4G_2. Grouped Frequency Data for Grouping G_2

LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is lngdset2.dat

		Moment Estimate		Max Likelihood Est	
v	f(abs)	f(cal)	X**2	f(cal)	X**2
1	8	7.56	.03	7.10	.11
2	18	18.90	.04	20.67	.35
3	25	21.69	.50	22.23	.35
4	16	16.76	.03	16.09	.00
5	7	10.41	1.11	9.70	.75
6	7	5.68	.30	5.34	.51
7	3	2.88	.01	2.82	.01
8	2	1.39	.27	1.46	.20
9	0	.66	.66	.75	.75
10	1	.31	1.58	.39	.97
Totals	87	86.23		86.55	

TABLE 1.4G_3. Grouped Frequency Data for Grouping G_3

LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is lngdset3.dat

		Moment Estimate		Max Likelihood Est	
v	f(abs)	f(cal)	X**2	f(cal)	X**2
1	6	3.62	1.56	3.29	2.23
2	5	7.13	.64	7.54	.85
3	8	10.00	.40	10.78	.72
4	15	11.33	1.19	11.94	.78
5	12	11.13	.07	11.36	.04
6	11	9.92	.12	9.82	.14
7	10	8.25	.37	7.98	.51
8	0	6.53	6.53	6.22	6.22
9	5	4.99	.00	4.71	.02
10	5	3.70	.45	3.50	.64
11	3	2.70	.03	2.57	.07
12	1	1.94	.45	1.87	.40
13	3	1.38	1.91	1.35	2.01
14	1	.97	.00	.98	.00
15	1	.68	.15	.70	.13
16	0	.48	.48	.51	.51
17	0	.33	.33	.37	.37
18	0	.23	.23	.26	.26
19	0	.16	.16	.19	.19
20	1	.11	6.97	.14	5.32
Totals	87	85.61		86.08	

TABLE 1.4G_4. Grouped Frequency Data for Grouping G_4

LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is lngdset4.dat

		Moment Estimate		Max Likelihood Est	
v	f(abs)	f(cal)	X**2	f(cal)	X**2
1	2	2.51	.10	1.88	.01
2	9	8.90	.00	9.19	.00
3	15	14.89	.00	15.85	.05
4	20	16.54	.72	17.03	.52
5	13	14.42	.14	14.28	.11
6	8	10.81	.73	10.41	.56
7	7	7.34	.02	6.98	.00
8	6	4.66	.38	4.45	.54
9	3	2.84	.01	2.75	.02
10	2	1.68	.06	1.67	.06
11	1	.97	.00	1.00	.00
12	0	.56	.56	.60	.60
13	0	.32	.32	.36	.36
14	1	.18	3.74	.22	2.84
Totals	87	86.62		86.65	

TABLE 1.4G_5. Grouped Frequency Data for Grouping G_5

LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is lngdset5.dat

		Moment Estimate		Max Likelihood Est	
v	f(abs)	f(cal)	X**2	f(cal)	X**2
1	6	4.37	.60	4.18	.79
2	8	12.35	1.53	12.53	1.64
3	20	16.99	.54	17.26	.44
4	17	16.31	.03	16.40	.02
5	16	12.83	.78	12.77	.82
6	4	8.97	2.76	8.87	2.68
7	6	5.84	.00	5.76	.01
8	4	3.64	.04	3.59	.05
9	3	2.21	.29	2.19	.30
10	1	1.32	.08	1.31	.07
11	1	.78	.06	.78	.06
12	0	.46	.46	.47	.47
13	0	.27	.27	.28	.28
14	1	.16	4.43	.17	4.18
Totals	87	86.49		86.56	

TABLE 1.5G_1. Three-Parameter Estimates for Grouping G_1

```
LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is  LNGDSET1.DAT
Number of parameters is   3

***************************************************************
THE MOMENT ESTIMATES FOR THE LOG-NORMAL DISTRIBUTION
Lambda(0)  =     -.838910E+002
Sigma(0)   =      .342081E+000
Mu(0)      =      .618104E+001

***************************************************************
THE DERIVATIVE ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(1)  =      .731777E+002
Sigma(1)   =      .505160E+000
Mu(1)      =      .574925E+001

***************************************************************
THE POWELL ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(2)  =      .731776E+002
Sigma(2)   =      .505160E+000
Mu(2)      =      .574925E+001

***************************************************************
```

TABLE 1.5G_2. Three-Parameter Estimates for Grouping G_2

```
LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is  LNGDSET2.DAT
Number of parameters is   3

***************************************************************
THE MOMENT ESTIMATES FOR THE LOG-NORMAL DISTRIBUTION
Lambda(0)  =     -.122396E+003
Sigma(0)   =      .317648E+000
Mu(0)      =      .625996E+001

***************************************************************
THE DERIVATIVE ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(1)  =     -.141964E+002
Sigma(1)   =      .398159E+000
Mu(1)      =      .601291E+001

***************************************************************
THE POWELL ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(2)  =     -.141964E+002
Sigma(2)   =      .398159E+000
Mu(2)      =      .601291E+001

***************************************************************
```

TABLE 1.5G3. Three-Parameter Estimates for Grouping G_3

```
 LOG-NORMAL DISTRIBUTION CALCULATIONS
 Data file is  LNGDSET3.DAT
Number of parameters is   3

*****************************************************
THE MOMENT ESTIMATES FOR THE LOG-NORMAL DISTRIBUTION
Lambda(0)  =     -.574054E+002
Sigma(0)   =      .357288E+000
Mu(0)      =      .611734E+001

*****************************************************
THE DERIVATIVE ESTIMATES FOR THE LOG-NORMAL DIST
Lambda( 1) =      .158447E+002
Sigma( 1)  =      .422920E+000
Mu( 1)     =      .592810E+001

*****************************************************
THE POWELL ESTIMATES FOR THE LOG-NORMAL DIST
Lambda( 2) =      .158448E+002
Sigma( 2)  =      .422920E+000
Mu( 2)     =      .592810E+001

*****************************************************
```

TABLE 1.5G4. Three-Parameter Estimates for Grouping G_4

```
 LOG-NORMAL DISTRIBUTION CALCULATIONS
 Data file is  LNGDSET4.DAT
Number of parameters is   3

*****************************************************
THE MOMENT ESTIMATES FOR THE LOG-NORMAL DISTRIBUTION
Lambda(0)  =     -.104731E+003
Sigma(0)   =      .326079E+000
Mu(0)      =      .622973E+001

*****************************************************
THE DERIVATIVE ESTIMATES FOR THE LOG-NORMAL DIST
Lambda( 1) =     -.265338E+002
Sigma( 1)  =      .382665E+000
Mu( 1)     =      .605208E+001

*****************************************************
THE POWELL ESTIMATES FOR THE LOG-NORMAL DIST
Lambda( 2) =     -.265343E+002
Sigma( 2)  =      .382665E+000
Mu( 2)     =      .605209E+001

*****************************************************
```

TABLE 1.5G$_5$. Three-Parameter Estimates for Grouping G$_5$

```
 LOG-NORMAL DISTRIBUTION CALCULATIONS
 Data file is  LNGDSETS.DAT
Number of parameters is   3

*****************************************************************
THE MOMENT ESTIMATES FOR THE LOG-NORMAL DISTRIBUTION
Lambda(0)  =     -.436832E+002
Sigma(0)   =      .370006E+000
Mu(0)      =      .607871E+001

*****************************************************************
THE DERIVATIVE ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(1)  =     -.246130E+002
Sigma(1)   =      .384999E+000
Mu(1)      =      .603142E+001

*****************************************************************
THE POWELL ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(2)  =     -.246162E+002
Sigma(2)   =      .384997E+000
Mu(2)      =      .603143E+001

*****************************************************************
```

TABLE 2.1. A Random Lognormal Sample

148.290	184.101	135.880	127.211
133.143	144.328	166.475	137.338
132.971	156.680	174.800	131.375
164.304	128.709	153.070	168.554
145.788	155.369	201.415	157.238

TABLE 2.2. Three-Parameter Estimates

```
 LOG-NORMAL DISTRIBUTION CALCULATIONS
 Data file is   RANDSAM.DAT
Number of parameters is    3

*****************************************************
THE MOMENT ESTIMATES FOR THE LOG-NORMAL DISTRIBUTION
Lambda(0)  =      .727727E+002
Sigma(0)   =      .241002E+000
Mu(0)      =      .434708E+001

*****************************************************
THE DERIVATIVE ESTIMATES FOR THE LOG-NORMAL DIST
Lambda( 1) =      .117721E+003
Sigma( 1)  =      .604313E+000
Mu( 1)     =      .337317E+001

*****************************************************
THE POWELL ESTIMATES FOR THE LOG-NORMAL DIST
Lambda( 2) =      .117784E+003
Sigma( 2)  =      .605899E+000
Mu( 2)     =      .337039E+001

*****************************************************
```

TABLE 2.3. Two-Parameter Estimates

```
 LOG-NORMAL DISTRIBUTION CALCULATIONS
 Data file is  RANOSAM.DAT
Number of parameters is   2

****************************************************
THE MOMENT ESTIMATES FOR THE LOG-NORMAL DISTRIBUTION
Lambda(0) =        .100000E+003
Sigma(0)  =        .127181E+000
Mu(0)     =        .501778E+001

****************************************************
THE DERIVATIVE ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(1) =        .100000E+003
Sigma(1)  =        .365203E+000
Mu(1)     =        .389015E+001

****************************************************
THE POWELL ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(2) =        .100000E+003
Sigma(2)  =        .365203E+000
Mu(2)     =        .389015E+001

****************************************************
```

TABLE 3.1. Number of Frost-Days in the Months of April, 1930-1966 and Their Frequencies

0	1	2	2	2	2	2	2
3	3	3	3	4	4	4	4
4	4	5	5	5	6	6	7
7	8	8	8	8	9	9	9
10	10	10	12	17			

TABLE 3.2. Three-Parameter Estimates

```
 LOG-NORMAL DISTRIBUTION CALCULATIONS
 Data file is  LNDSET4.DAT
Number of parameters is   3

****************************************************
THE MOMENT ESTIMATES FOR THE LOG-NORMAL DISTRIBUTION
Lambda(0)  =      -.639374E+001
Sigma(0)   =       .288822E+000
Mu(0)      =       .244448E+001

****************************************************
THE DERIVATIVE ESTIMATES FOR THE LOG-NORMAL DIST
Lambda( 1) =      -.302085E+001
Sigma( 1)  =       .407262E+000
Mu( 1)     =       .207493E+001

****************************************************
THE POWELL ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(2)  =      -.302085E+001
Sigma(2)   =       .407262E+000
Mu(2)      =       .207493E+001

****************************************************
```

TABLE 4.1. Rainfall Totals (in Inch) for Sets of Four Consecutive Months, Camden Square, London, 1870-1943

LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is brooks.dat

		Moment Estimate		Max Likelihood Est	
v	f(abs)	f(cal)	X**2	f(cal)	X**2
1	9	4.77	3.75	4.69	3.97
2	15	19.92	1.21	19.78	1.16
3	48	50.43	.12	50.35	.11
4	81	89.11	.74	89.19	.75
5	142	120.66	3.77	120.85	3.70
6	125	133.54	.55	133.73	.57
7	129	126.52	.05	126.63	.04
8	109	106.15	.08	106.16	.08
9	76	80.89	.30	80.84	.29
10	56	57.07	.02	57.01	.02
11	36	37.86	.09	37.80	.09
12	22	23.88	.15	23.84	.14
13	13	14.46	.15	14.44	.14
14	12	8.47	1.47	8.46	1.48
15	5	4.83	.01	4.83	.01
16	4	2.69	.63	2.69	.63
17	3	1.48	1.57	1.48	1.57
Totals	885	882.73		882.75	

TABLE 4.2. Three-Parameter Estimates

```
 LOG-NORMAL DISTRIBUTION CALCULATIONS
 Data file is  BROOKS.DAT
Number of parameters is   3

**************************************************************
THE MOMENT ESTIMATES FOR THE LOG-NORMAL DISTRIBUTION
Lambda(0)  =     -.438818E+001
Sigma(0)   =      .215054E+000
Mu(0)      =      .253126E+001

**************************************************************
THE DERIVATIVE ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(1)  =     -.431653E+001
Sigma(1)   =      .216064E+000
Mu(1)      =      .252544E+001

**************************************************************
THE POWELL ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(2)  =     -.431654E+001
Sigma(2)   =      .216064E+000
Mu(2)      =      .252544E+001

**************************************************************
```

TABLE 4.3. Two-Parameter Estimates

```
LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is  BROOKS.DAT
Number of parameters is   2

***************************************************
THE MOMENT ESTIMATES FOR THE LOG-NORMAL DISTRIBUTION
Lambda( 0)  =        .000000E+000
Sigma( 0)   =        .321706E+000
Mu( 0)      =        .208539E+001

***************************************************
THE DERIVATIVE ESTIMATES FOR THE LOG-NORMAL DIST
Lambda( 1)  =        .000000E+000
Sigma( 1)   =        .341545E+000
Mu( 1)      =        .208132E+001

***************************************************
THE POWELL ESTIMATES FOR THE LOG-NORMAL DIST
Lambda( 2)  =        .000000E+000
Sigma( 2)   =        .341545E+000
Mu( 2)      =        .208132E+001

***************************************************
```

TABLE 4.4. Expected Frequencies for Two-Parameter Estimates

LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is brooks.dat

		Moment Estimate		Max Likelihood Est	
v	f(abs)	f(cal)	X**2	f(cal)	X**2
1	9	.58	122.01	1.23	49.09
2	15	10.88	1.56	15.57	.02
3	48	47.45	.01	55.07	.91
4	81	99.01	3.28	102.35	4.45
5	142	135.60	.30	131.76	.80
6	125	143.13	2.30	135.25	.78
7	129	127.51	.02	119.83	.70
8	109	101.29	.59	96.14	1.72
9	76	74.31	.04	72.02	.22
10	56	51.54	.39	51.41	.41
11	36	34.35	.08	35.47	.01
12	22	22.24	.00	23.88	.15
13	13	14.12	.09	15.81	.50
14	12	8.83	1.14	10.34	.27
15	5	5.47	.04	6.71	.43
16	4	3.37	.12	4.33	.02
17	3	2.06	.43	2.79	.02
Totals	885	881.73		879.95	

TABLE 5.1. Weekly Precipitation Sums (in 10^{-2} inch), Kwajalein, IS, 1949-1958

LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is fq.kwa

		Moment Estimate		Max Likelihood Est	
v	f(abs)	f(cal)	X**2	f(cal)	X**2
1	8	10.20	.48	8.84	.08
2	25	15.68	5.54	22.26	.34
3	21	18.08	.47	23.01	.18
4	15	17.58	.38	18.98	.84
5	18	15.36	.45	14.48	.86
6	9	12.51	.99	10.73	.28
7	8	9.73	.31	7.69	.00
8	6	7.33	.24	5.81	.01
9	5	5.40	.03	4.30	.12
10	4	3.92	.00	3.20	.20
11	3	2.82	.01	2.40	.15
12	1	2.02	.51	1.82	.37
13	1	1.43	.13	1.39	.11
14	1	1.02	.00	1.07	.00
15	2	.72	2.25	.83	1.64
16	2	.52	4.28	.65	2.80
17	1	.37	1.09	.51	.46
Totals	130	124.68		128.19	

TABLE 5.2. Three-Parameter Estimates

```
 LOG-NORMAL DISTRIBUTION CALCULATIONS
 Data file is  FQ.KWA
Number of parameters is   3

************************************************************
THE MOMENT ESTIMATES FOR THE LOG-NORMAL DISTRIBUTION
Lambda(0)  =     -.180499E+003
Sigma(0)   =      .410379E+000
Mu(0)      =      .593504E+001

************************************************************
THE DERIVATIVE ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(1)  =     -.216921E+002
Sigma(1)   =      .694260E+000
Mu(1)      =      .530061E+001

************************************************************
THE POWELL ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(2)  =     -.216921E+002
Sigma(2)   =      .694260E+000
Mu(2)      =      .530061E+001

************************************************************
```

TABLE 5.3. Two-Parameter Estimates

```
LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is  FQ.KWA
Number of parameters is   2

****************************************************
THE MOMENT ESTIMATES FOR THE LOG-NORMAL DISTRIBUTION
Lambda(0)  =      .000000E+000
Sigma(0)   =      .677532E+000
Mu(0)      =      .521189E+001

****************************************************
THE DERIVATIVE ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(1)  =      .000000E+000
Sigma(1)   =      .819763E+000
Mu(1)      =      .514332E+001

****************************************************
THE POWELL ESTIMATES FOR THE LOG-NORMAL DIST
Lambda(2)  =      .000000E+000
Sigma(2)   =      .819763E+000
Mu(2)      =      .514332E+001

****************************************************
```

TABLE 5.4. Expected Frequencies for Two-Parameter Estimates

LOG-NORMAL DISTRIBUTION CALCULATIONS
Data file is fq.kwa

		Moment Estimate		Max Likelihood Est	
v	f(abs)	f(cal)	X**2	f(cal)	X**2
1	8	2.02	17.66	8.04	.00
2	25	21.35	.62	25.39	.01
3	21	26.08	.99	23.51	.27
4	15	21.82	2.13	18.07	.52
5	18	16.25	.19	13.30	1.66
6	9	11.64	.60	9.74	.06
7	8	8.25	.01	7.17	.10
8	6	5.85	.00	5.34	.08
9	5	4.17	.16	4.03	.24
10	4	3.01	.33	3.07	.28
11	3	2.19	.30	2.37	.17
12	1	1.61	.23	1.85	.39
13	1	1.19	.03	1.45	.14
14	1	.89	.01	1.16	.02
15	2	.67	2.60	.93	1.24
16	2	.51	4.29	.75	2.09
17	1	.40	.92	.61	.25
Totals	130	127.91		126.77	

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